Chapter 2
Chaos theory and its relationship to complexity

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This chapter introduces chaos theory and the concept of non-linearity. It highlights the importance of reiteration and the system features that arise from it. Although the relationship between chaos and complexity remains contested, chaos theory offers a useful starting point to understand complex systems.

Key points

- Chaos theory is the quantitative study of dynamic non-linear system.
- Non-linear systems change with time and can demonstrate complex relationships between inputs and outputs due to reiterative feedback loops within the system.
- Providing sufficient computational power is available, these systems are predictable but their behaviour is exquisitely sensitive to their starting point
- Due to the complex nature of social systems, the mathematical application of chaos theory is limited to all but the simplest of systems.
- The relationship between chaos and complexity is contested. A useful starting point from an organisational perspective is to see complexity theory as the qualitative study of non-linear systems drawing its metaphors from chaos theory.
- Health systems are non-linear. This has profound implications for analysis, prediction and control.

Introduction

The relationship between chaos and complexity theories is contested which is not a useful start. The range of opinion includes: chaos is a sub-discipline of complexity; chaos and complexity are interchangeable and the distinction is arbitrary; the two phenomenon have different origins and should not be considered together; the "zone of complexity" sits at "edge of chaos"; the study of chaos is unhelpful and should be ignored. From an organisational perspective, I find the following model a useful starting point and one that makes sense to me:

Chaos and complexity theory studies dynamic non-linear systems i.e. systems that change with time and demonstrate complex relationships between inputs and outputs due to reiterative feedback loops within the system. The quantitative study of these systems is *chaos theory. Complexity theory is the qualitative aspect drawing upon insights and metaphors that are derived from chaos theory.*

This book adopts that approach and this chapter explores the insights that chaos theory offers on the behaviour of non-linear systems.

What is non-linearity?

A non-linear system is one in which there is no simple relationship between cause and effect[1]. The main characteristics of non-linearity are:

- Small inputs can have large system effects
- Large inputs can lead to small system changes
- There is extreme sensitivity of behaviour to initial conditions. Small changes in a variable in the system at one point will make a very large difference in the behaviour of a system at some future point. When error is introduced into a linear system the prediction error stays relatively constant over time. For non-linear systems prediction error increases rapidly with time. Due to our inability to measure initial system conditions accurately and the extreme sensitivity of non-linear systems behaviour to these initial conditions, in practical terms, most natural non-linear systems are unpredictable.

What is chaos?

Unfortunately, the term chaos is a misnomer and confusion arises from the outset over the common and mathematical interpretations of the term[2]. Chaotic behaviour appears random but when studied in a particular way, ordered features and patterns are discernible. Chaos can be understood by comparing it with two other types of behaviours – randomness and periodicity. These are shown in Figure 1.

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[1] A variable changing in time in a linear manner allows simple prediction by adding a weighed sum of previous observations. A non-linear time series violates this simple assumption by including squares, cubes, etc of previous terms or some more complex transformation such as the exponential. Prediction is still possible but needs much greater computational power. In more formal terms:

Let x_1 and y_1 be respectively the input and output of a linear system λ. Let x_2 and y_2 be another pair of input and output of the same system λ. The output *y* corresponding to the combined linear sum of both inputs is proportional to the linear sum of the outputs corresponding to each of the inputs separately. In other words, if $y_1 = \lambda(x_1)$ and $y_2 = \lambda(x_2)$, then $y = \lambda(\alpha x_1 + \beta x_2) = \alpha y_1 + \beta y_2$ where α and β are constants. In the case of non-linear systems, this does not hold.

[2] In the technical sense, the term chaos is used to denote a form of time evolution in dynamic systems in which the difference between two states that are initially very close grows exponentially with time. The Lyapunov exponent is a measure of this divergence and can be used to quantify chaotic systems.

Chaos can be understood by comparing it with two other types of behaviour – randomness and periodicity.

Random behaviour never repeats itself although we can predict the average behaviour of a system using statistics.

Periodic behaviour is highly predictable because it always repeats itself, for example, the swinging of a pendulum. Such systems are deterministic, ie if we know the conditions at any one point we can predict those conditions at any other point in time.

Chaos has characteristics of both behaviours. Although it looks disorganised like random behaviour, it is deterministic like periodic behaviour. However, the smallest difference in any system variable can make a very large difference to the future state of the system

Figure 1 - what is chaos?

Chaotic systems are characterised by three key properties: predictability, extreme sensitivity to initial conditions and presence of an attractor or pattern of behaviour. Chaotic patterns form the signature of non-linear behaviour that arises from recursive feedback among a system's components i.e. the output of one stage feeds back into the input of the next. (This recursive or re-iterative feature is critical to complex systems as it sets the focus of attention at a local level.) There are a number of approaches both graphical and numerical beyond the scope of this text to decide whether a system is chaotic or not. It is important to note that non-linear systems are not necessarily chaotic but non-linearity is prerequisite to chaos.

Although chaotic behaviour was suspected over a hundred years ago, it has only been the availability of computational power that has enabled scientists to probe the complex mathematical interior of non-linear equations. Over the last decade, the suspicion that chaos may play an important role in the functioning of living systems has been confirmed[1 2]. It seems that chaos is the healthy signature of physiology and during abnormal conditions such as a heart attack, systems revert to non-chaotic behaviour. The publication of James Gleick, "Chaos: making a new science"[3] alerted a wider audience to the importance of an area that has now found applications as widespread as the study of the weather to the behaviour of stock markets.

In the next section, the concept of non-linearity and the road to chaotic behaviour is explored in more detail.

Non linearity and the road to Chaos

We first explore a basic non-linear equation. To do so we use a model of the population of fish in a pool year on year. Where n is the number of fish in the pool in a given year and B is the birth rate, equation 1 shows a linear model to describe and predict events. (Although this equation describes an exponential curve, it is linear. Linear does not mean a straight line!)

Equation 1:

$$n_{\text{(next year)}} = B.n_{\text{(this year)}}.$$

However, a more realistic model is in shown by equation 2, where n_{max} is the maximum number of fish that the pond can accommodate. As the fish numbers increase, the food supply reduces and the term $(n_{max} - n_{\text{(this year)}})$ is introduced as a reiterative feedback term.

Equation 2:

$$n_{\text{(next year)}} = B.n_{\text{(this year)}}.\ (n_{max} - n_{\text{(this year)}})$$

This introduces a non-linear term into the equation. (This simple feedback equation could be a the starting point to explain behaviour in health systems. For example, I refer patients to the physiotherapy department but as the waiting list lengthens my referral rate is reduced. As patients are seen quicker, I refer more patients.)

We explore what happens year on year as we vary the birth B. To keep the sums simple, we take the maximum value of n as 1 and all values of n as proportions of 1.

Figure 2 shows a record of the fish number that will be found in the pond for a given value of B.

INSERT FIG 2 HERE - SEE DOC END

For values of B under 3, the fish population converges or is attracted to a constant population (this is called a point attractor). However, when B reaches a value of 3 this attractor becomes unstable and splits or bifurcates into two - the population oscillates year on year around two stable values (a periodic attractor). (Bifurcation is the point in which there is an abrupt change in behaviour of a dynamic system that occurs when one of the parameters reaches a critical value.) As we increase the birth rate, these points split again until at a level of B of 3.57, plotting the fish in the pond year on year generates a huge number of values. We have arrived at chaotic behaviour. Although this pattern looks random, for each birth rate we look at, the values of population that we can get will generate a geometric pattern around what is known as a chaotic or strange attractor. We will explore this in a little more detail shortly. The pattern is also boundable. We can describe a "possibility space" in which the solutions to the equation can be found.

Another interesting feature of this patterning is self-similarity at different levels of scale. If we examine a very small portion of the graph and amplify it we will see the pattern repeating itself at smaller and smaller levels of scale. This is known as a fractal phenomena[3].

From our exploration, we can draw some important conclusions about non-linear behaviour:

- Complex behaviour can arise from the re-iterative application of very simple equations or rules.
- Non-linear systems are predictable, providing we have adequate computational power to undertake the re-iterative calculations and accurate starting conditions.
- Non-linear systems are exquisitely sensitive to where they start from (their initial conditions). For example, changing the value of B only very slightly can dramatically alter the output of the system. In practice, it is this feature that makes non-linear systems so unpredictable - we can never measure their initial conditions with absolute certainty.

If organisations demonstrate non-linear characteristics we can make some interesting postulations:

- Our ability to predict events and engineer the system towards a defined objective may be limited. Any predictability will be short-term due to the rapidly cumulative effects of feedback.
- Initial conditions are important. What happens in organisations will be influenced by what has gone before - you have to know where you have been to see where you might go.
- We can expect to see patterns re-occurring at different levels of system scale.
- The recursive application of a few simple rules may lead to complex organisational behaviour.
- Non-linear interaction between individuals will modulate their differences and create novelty that may not have been anticipated.

In the next section we explore two further concepts of chaos theory - phase space and attractors.

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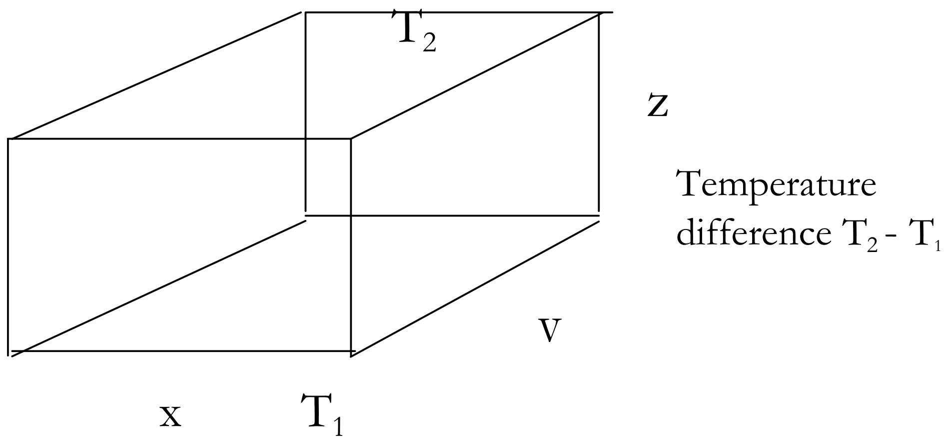
[3] The fractal dimension is defined as the slope of the function relating the numbers of points contained in a given "magnification" to the magnification itself.

Phase space and attractors

One way of describing a dynamic system is by plotting its trajectory with time. If we describe an element in the system using *n* variables and for each variable allocate one dimension on a graph, we can plot the trajectory of that element in an *n* dimensional graph or phase space. This is the space that contains the range of values that can be found in a particular system.

Figure 3 shows a simple model to describe the motion of a particle in a fluid across which there is a temperature gradient. It consists of three inter-related equations that form a non-linear system. The output can be drawn in 3-dimensional phase space as shown in Figure 4. We can see it forms a trajectory around a particular area of phase space - a chaotic attractor. The attractor is an area in phase space where the trajectories are more likely to be found. Although in theory we can calculate the exact position of the particle at any point in time, its trajectory will be extremely sensitive to its initial starting position. However, we can be more certain about the pattern that will be described.



dx/dt = P(y-x)

dy/dt = R x-y-xy

dz/dt - xy - By

Where P, R and B are constants

Fig 3 Non-linear equations to describe motion of a particle in a fluid across which there is a temperature difference

INSERT FIG 4 HERE - SEE DOC END

We can use these concepts as metaphors when describing non-linear systems qualitatively. For example, in organisations we follow trajectories in phase space around attractors or system fundamentals. In the NHS one important attractor could be equity. In the US health system, profit would play a more important role.

Conclusion

Chaos theory studies non-linear systems mathematically. Although there have been advances in the prediction and control of the trajectories of chaotic systems,[4 5] these have been from a theoretical mathematical perspective and it is unlikely that these developments could be extrapolated to human systems. However, chaos can provide useful insights and metaphors for understanding social organisations that challenge much of current thinking.

From our own observations, we might deduce that the health system is non-linear. But more formal evidence is forthcoming. For example, Papadopoulos[6] analysed surgical waiting lists in the NHS and confirmed chaotic properties. It was also suggested that waiting lists also demonstrate fractal properties i.e. similar chaotic structure could be identified not only on waiting lists by speciality but on the waiting lists of individual consultants. The conclusion was that government measures to reduce waiting lists were destined to failure!

However, in most areas of organisational life mathematical analysis becomes restricted - the algorithms or equations that describe our interaction (our mental models that determine how we respond to the environment) are continually changing as we interact and learn. Never the less, we can gain some useful insights applying chaos principles to social organisations within the framework known as complexity theory. It is to this we turn in the next chapter.

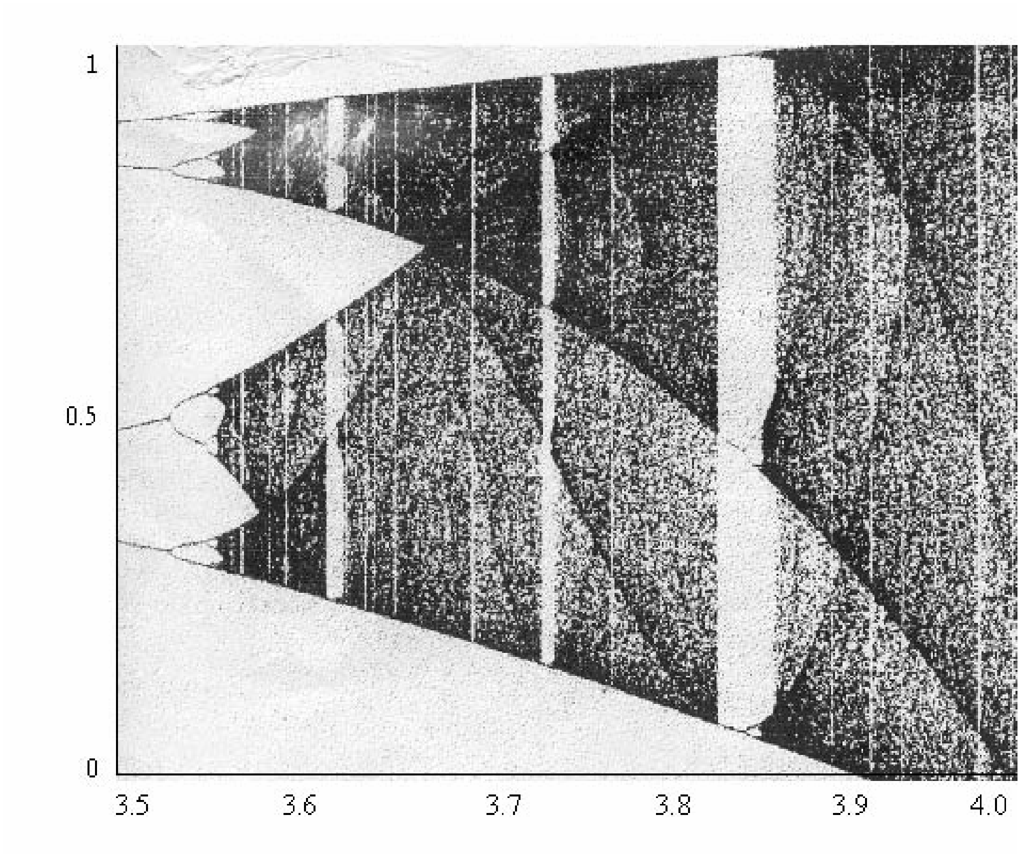


Figure 2 - numbers of fish found in pool 'n' in successive years with different values of birth rate B

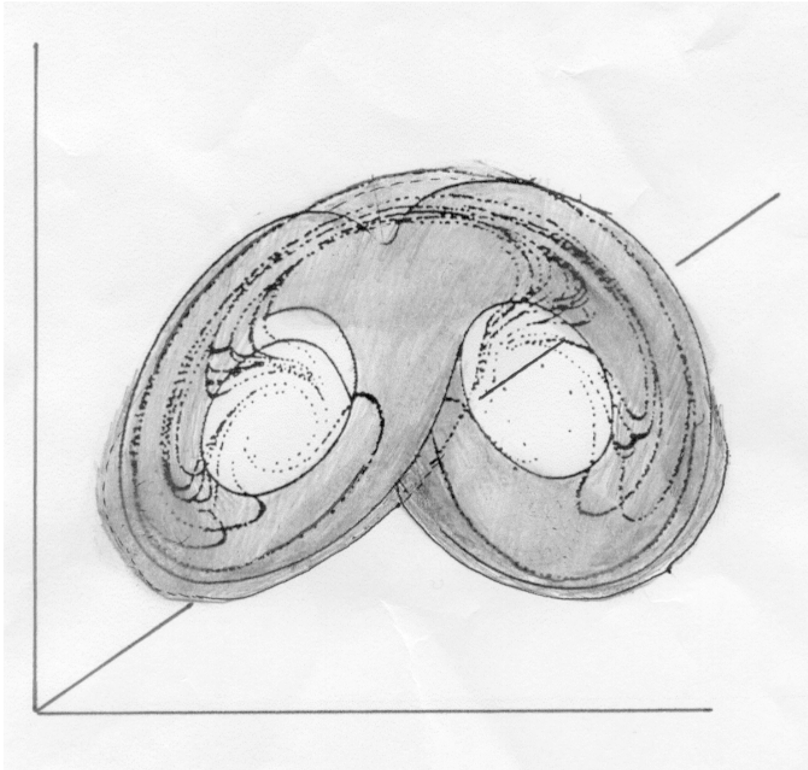


Figure 4. The path of a particle around an attractor in three dimensional phase space calculated from the non-linear model shown in figure 2

1 Goldberger A. Non-linear dynamics for clinicians: chaos theory, fractals and complexity at the bedside. *Lancet* 1996;347:1312-14.

2 Sataloff R, Hawkshaw M. Chaos and medicine - source readings. *Singular Press:* San Diego, 2001.

3 Gleick J. Chaos: making a new science. *Penguin Books:* London, 1987.

4 Kantz H, Schreiber T. Nonlinear time series analysis. Cambridge University Press: Cambridge, 1997.

5 Kapitaniak T, Controlling chaos. Theoretical and practical methods in non-linear dynamics. Academic Press: London, 1996.

6 Papadopoulos M, Hadjitheodossiou M, Chrysostomu C, et al. Is the National Health Service at the edge of chaos? *Journal of the Royal Society of Medicine* 2001; 94(12): 613-16.